

# Calculating $\lim_{x \rightarrow a} f(x)$

Suppose that  $c$  is a constant and the limits  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exist. Then

**Sum Law**  $\lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$

**Difference Law**  $\lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$

**Constant Multiple Law**  $\lim_{x \rightarrow a} (c f(x)) = c \lim_{x \rightarrow a} f(x)$

**Product Law**  $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

**Quotient Law**  $\lim_{x \rightarrow a} \left( \frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$

**Power Law**  $\lim_{x \rightarrow a} (f(x))^n = \left( \lim_{x \rightarrow a} f(x) \right)^n$

**Root Law**  $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ , where  $n$  is a positive integer

**Constant Law**  $\lim_{x \rightarrow a} c = c$

**Direct Substitution Law**  $\lim_{x \rightarrow a} f(x) = f(a)$

## L'Hospital's Rule

If  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$  or  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\pm \infty}{\pm \infty}$  then,

$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$   $a$  is a number,  $\infty$  or  $-\infty$

Find  $\lim_{x \rightarrow -2} \frac{x+2}{x^2+3x+2}$

(step 1) Plug in to evaluate the limit, if possible

$\lim_{x \rightarrow -2} \frac{x+2}{x^2+3x+2} = \frac{(-2)+2}{(-2)^2+3(-2)+2} = \frac{0}{0}$  (indeterminate)

(step 2) Apply L'Hospital's rule and reevaluate the limit

$\lim_{x \rightarrow -2} \frac{\frac{d}{dx}[x+2]}{\frac{d}{dx}[x^2+3x+2]} = \lim_{x \rightarrow -2} \frac{1}{2x+3}$   
 $\lim_{x \rightarrow -2} \frac{1}{2x+3} = \frac{1}{2(-2)+3} = \frac{1}{-1} = \boxed{-1}$

**critical point:**  $f'(c)=0$  or DNE

**incr/decr:**

- 1) find crit pts of  $f$
- 2) make sign chart for  $f'$
- 3) plug in values to determine if incr or decr

**find abs max/min of  $f$  on  $[a,b]$**

- 1) evaluate  $f$  @  $x = a, b$  (endpoints)
- 2) find crit pts on  $[a,b]$  (set  $der=0$ ) & evaluate  $f$  @ crit pts
- 3) compare values (largest=abs max; smallest=abs min)

**concavity:**

- 1)  $f''(x)$
- 2) sign chart for  $f''$
- 3) test for concavity
- 4) inflection pt = where concavity switches

$\int x^2(x^3-7)^3 dx$

$\int x^2 u^3 dx$  **Substitute  $u$**

$\int x^2 u^3 \frac{1}{3x^2} du$  **Substitute  $du$**

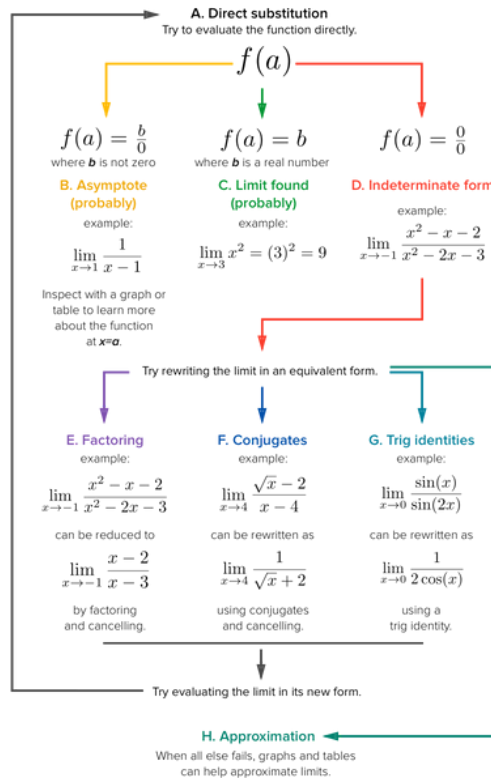
$\int \frac{1}{3} u^3 du$  **Cancel the  $x^2$**

$\frac{1}{3} \int u^3 du$  **Factor out the  $1/3$**

**$u$  Substitution:** The substitution  $u = g(x)$  will convert  $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u) du$  using

$du = g'(x)dx$ . For indefinite integrals drop the limits of integration.

Ex.  $\int_1^2 5x^2 \cos(x^3) dx$   $\int_1^2 5x^2 \cos(x^3) dx = \int_1^8 \frac{5}{3} \cos(u) du$   
 $u = x^3 \Rightarrow du = 3x^2 dx \Rightarrow x^2 dx = \frac{1}{3} du$   $= \frac{5}{3} \sin(u) \Big|_1^8 = \frac{5}{3} (\sin(8) - \sin(1))$   
 $x = 1 \Rightarrow u = 1^3 = 1 \therefore x = 2 \Rightarrow u = 2^3 = 8$



Differentiation Rules	
<b>Constant Rule</b>	$\frac{d}{dx}[c] = 0$
<b>Power Rule</b>	$\frac{d}{dx}x^n = nx^{n-1}$
<b>Product Rule</b>	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
<b>Quotient Rule</b>	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
<b>Chain Rule</b>	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

## Derivative

$\frac{d}{dx}n = 0$

$\frac{d}{dx}x = 1$

$\frac{d}{dx}x^n = nx^{n-1}$

$\frac{d}{dx}e^x = e^x$

$\frac{d}{dx}\ln x = \frac{1}{x}$

$\frac{d}{dx}n^x = n^x \ln n$

$\frac{d}{dx}\sin x = \cos x$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x$

$\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$\frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}$

$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{x\sqrt{x^2-1}}$

$\frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{x\sqrt{x^2-1}}$

## Integral (Antiderivative)

$\int 0 dx = C$

$\int 1 dx = x + C$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$\int e^x dx = e^x + C$

$\int \frac{1}{x} dx = \ln x + C$

$\int n^x dx = \frac{n^x}{\ln n} + C$

$\int \cos x dx = \sin x + C$

$\int \sin x dx = -\cos x + C$

$\int \sec^2 x dx = \tan x + C$

$\int \csc^2 x dx = -\cot x + C$

$\int \tan x \sec x dx = \sec x + C$

$\int \cot x \csc x dx = -\csc x + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

$\int -\frac{1}{\sqrt{1-x^2}} dx = \arccos x + C$

$\int \frac{1}{1+x^2} dx = \arctan x + C$

$\int -\frac{1}{1+x^2} dx = \operatorname{arccot} x + C$

$\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C$

$\int -\frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arccsc} x + C$

## Chain Rule Variants

The chain rule applied to some specific functions.

1.  $\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1} f'(x)$
2.  $\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$
3.  $\frac{d}{dx}(\ln[f(x)]) = \frac{f'(x)}{f(x)}$
4.  $\frac{d}{dx}(\sin[f(x)]) = f'(x)\cos[f(x)]$
5.  $\frac{d}{dx}(\cos[f(x)]) = -f'(x)\sin[f(x)]$
6.  $\frac{d}{dx}(\tan[f(x)]) = f'(x)\sec^2[f(x)]$
7.  $\frac{d}{dx}(\sec[f(x)]) = f'(x)\sec[f(x)]\tan[f(x)]$
8.  $\frac{d}{dx}(\tan^{-1}[f(x)]) = \frac{f'(x)}{1+[f(x)]^2}$

Find the linearization of the function  $f(x) = \sqrt[3]{x}$  at  $a = -8$  and use it to approximate the number  $-8.1$

1. Plug in  $a = -8$  for  $x$  and solve for  $y$  to find ordered pair

$f(a) = \sqrt[3]{-8}$   
 $f(-8) = \sqrt[3]{-8} = -2$   
 $(-8, -2)$

2. Take the derivative to find the slope of the tangent line

$f(x) = \sqrt[3]{x} = x^{1/3}$   
 $\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

3. Plug in the ordered pair from step 1 and solve for the slope:  $\frac{dy}{dx} \Big|_{x=-8, y=-2}$

$\frac{dy}{dx} = \frac{1}{3(-8)^{2/3}}$   
 $m = \frac{1}{3(-2)^2/3} = \frac{1}{3(-2)^2}$   
 $m = \frac{1}{12}$

4. The Linearization is found by substituting the ordered pair and slope found in step 3 into Point-Slope Form

$y - y_1 = m(x - x_1)$   
 $y - (-2) = \frac{1}{12}(x - (-8))$   
 $y + 2 = \frac{1}{12}(x + 8)$   
 $y = \frac{1}{12}x - \frac{4}{3}$

5. Find the Linear Approximation of the number  $-8.1$ , plug it into the equation of the tangent line

$y = \frac{1}{12}x - \frac{4}{3}$   
 $y = \frac{1}{12}(-8.1) - \frac{4}{3}$   
 $y = -2.008$

## REVIEW

### The Limit Equation for Riemann's Sum

Upper limit of summation:  
It tells us to end with  $k = n$ .

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

Index of summation

Lower limit of summation:  
It tells us to start with  $k = 1$ .

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n (\Delta x) [f(x_k)]$$

It is more traditional to use  $k = 0$  for left or midpoint sums, and  $k = 1$  for right sums.

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§4.2b: Limits of Riemann's Sum

$$\Delta x = \frac{b-a}{n}$$

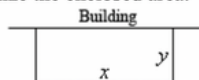
$$x_k = a + (\Delta x)k$$

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x,$$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^n f(x_i)\Delta x,$$

$$M_n = f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \cdots + f(\bar{x}_n)\Delta x = \sum_{i=1}^n f(\bar{x}_i)\Delta x,$$

**Ex.** We're enclosing a rectangular field with 500 ft of fence material and one side of the field is a building. Determine dimensions that will maximize the enclosed area.



Maximize  $A = xy$  subject to constraint of  $x + 2y = 500$ . Solve constraint for  $x$  and plug into area.

$$x = 500 - 2y \Rightarrow A = y(500 - 2y) = 500y - 2y^2$$

Differentiate and find critical point(s).  
 $A' = 500 - 4y \Rightarrow y = 125$

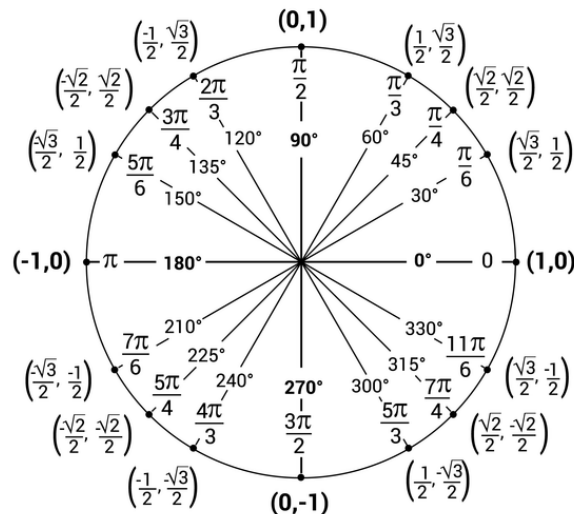
By 2<sup>nd</sup> deriv. test this is a rel. max. and so is the answer we're after. Finally, find  $x$ .

$$x = 500 - 2(125) = 250$$

The dimensions are then 250 x 125.

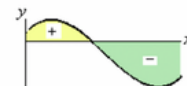
### optimization

- 1) draw diagram & introduce variables
- 2) write down conditions on the variables
- 3) determine the quantity to be maximized/minimized
- 4) write down an equation for the quantity to max/min & reduce # of variables
- 5) use der to find max/min value



### Applications of Integrals

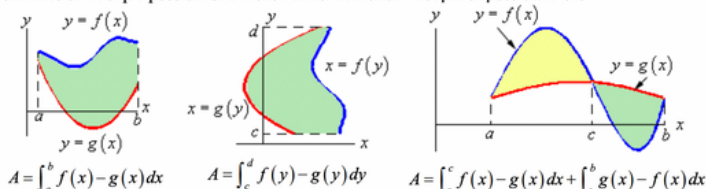
**Net Area :**  $\int_a^b f(x)dx$  represents the net area between  $f(x)$  and the  $x$ -axis with area above  $x$ -axis positive and area below  $x$ -axis negative.



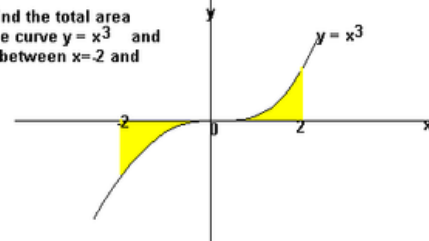
**Area Between Curves :** The general formulas for the two main cases for each are,

$$y = f(x) \Rightarrow A = \int_a^b [\text{upper function}] - [\text{lower function}] dx \quad \& \quad x = f(y) \Rightarrow A = \int_c^d [\text{right function}] - [\text{left function}] dy$$

If the curves intersect then the area of each portion must be found individually. Here are some sketches of a couple possible situations and formulas for a couple of possible cases.



**Example:** find the total area between the curve  $y = x^3$  and the  $x$ -axis between  $x = -2$  and  $x = 2$ .



If we simply integrated  $x^3$  between  $-2$  and  $2$ , we would get:

$$\left[ \frac{x^4}{4} \right]_{-2}^2 = 4 - 4 = 0$$

So instead, we have to split the graph up and do two separate integrals.

$$\int_0^2 x^3 dx = \left[ \frac{x^4}{4} \right]_0^2 = 16/4 - 0 = 4$$

$$\int_{-2}^0 x^3 dx = \left[ \frac{x^4}{4} \right]_{-2}^0 = 0 - 16/4 = -4 \quad (\text{so area is } 4).$$

We then add these two up to get: 8 units<sup>2</sup>

## RULES FOR TRANSFORMATIONS OF FUNCTIONS

If  $f(x)$  is the original function,  $a > 0$  and  $c > 0$ :

Function	Transformation of the graph of $f(x)$
$f(x) + c$	Shift $f(x)$ upward $c$ units
$f(x) - c$	Shift $f(x)$ downward $c$ units
$f(x + c)$	Shift $f(x)$ to the left $c$ units
$f(x - c)$	Shift $f(x)$ to the right $c$ units
$-f(x)$	Reflect $f(x)$ in the $x$ -axis
$f(-x)$	Reflect $f(x)$ in the $y$ -axis
$a \cdot f(x)$ , $a > 1$	Stretch $f(x)$ vertically by a factor of $a$ .
$a \cdot f(x)$ , $0 < a < 1$	Shrink $f(x)$ vertically by a factor of $a$ .
$f(ax)$ , $a > 1$	Shrink $f(x)$ horizontally by a factor of $\frac{1}{a}$ .
$f(ax)$ , $0 < a < 1$	Stretch $f(x)$ horizontally by a factor of $\frac{1}{a}$ .